# Short and long-term patterns of invention as a process of cultural accumulation

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## Abstract

The processes by which inventions are combinatorially generated and then gradually added to existing cultural traits across many generations is considered to be a fundamental driver of cultural evolution and technological change in human societies. Does the patent record provide evidence that modern invention is an instance of cultural accumulation? Here we argue that the presence of memory in the time series of patents in the short term and a transition from sub-exponential to super-exponential growth in the long term, constitute convincing evidence that patents accumulate through a process of collective learning akin to cultural accumulation. We propose a simple model of invention based on a process of recombination of old with current ideas that is consistent with data and other results from the literature. The results and simple modeling framework presented here highlight that although the pace and intensity of invention has accelerated in the modern era (*i.e.*, since the Industrial Revolution), *cultural accumulation* remains a relevant process in human invention connecting inventive activity across the whole of the history of *Homo Sapiens Sapiens*.

## 1 Introduction

What is the evidence that patents are indeed an instrument for fostering collective learning, as the builders of the U.S. patenting system intended them to be (1)? The patent system is designed to solve the problem of insufficient incentives to invent because of the high costs involved and the high risks of low rewards (due to the non-rivalry of ideas). By giving property and excludability rights to the inventor, the system is supposed to make invention (i) a profitable activity despite the costs and risks while, at the same time, (ii) create a self-propelled, self-referential, growing ecosystem of free-flowing ideas. There is not much debate about the former. The evidence for the latter, however, is less clear.

There has been extensive work investigating the flow of ideas (i.e., knowledge spillovers) across inventors, regions, and technologies (e.g., (2; 3)). This work has contributed to our understanding for how horizontal transfers of knowledge happen, and how they are typically mediated by proximity in space (4). Thus, there is sound cross-sectional evidence that the generation of ideas is a social and collective effort (5; 6). However, longitudinal evidence has been limited to case studies in the form of patent citations which, while suggestive, is neither necessary nor sufficient evidence that the patenting system fosters collective learning.

*Collective learning* refers to our species' defining ability to accumulate more knowledge with each passing generation than is lost by the next, each generating novelty and learning from what has come before (7; 8). Collective learning is thus a temporal process, but one in which accumulated knowledge both constrains and propels the future.

Culture is the example par excellence of a body of knowledge accumulated through many generations, and cultural accumulation is thus the preeminent archetype of collective learning. Humans owe their ecological success to culture (8). This cultural capacity enables humans to gradually accumulate information across generations in the form of tools, ideas, and practices that no individual could invent, or recreate, on their own (9). If invention—defined here as the creation of a new artifact, material, product, process, or algorithm which was otherwise not previously known in order to solve a problem—is a process of cultural accumulation, the patenting system stands as an institution that facilitates what is a basic process for humans to generate novelty (10). What evidence would refute or support this view?

As has now become common practice in the study of invention and innovation, we treat patents as a proxy measure for invention (11) and use the accumulation of patents in time as our main variable of interest. We will argue that the phenomena of inventors patenting in teams and patents citing previous patents both support the hypothesis that invention is a cultural process of collective learning. They are not, by themselves, direct evidence of cultural accumulation but rather evidence that there exists a channel through which cultural accumulation *might* have occurred. It is precisely these two features of patenting (collaboration and citations to "previous art") which make the question about whether invention is an instance of cultural accumulation worth asking.

We argue here that evidence for the presence of cultural accumulation must take the form of clear statistical patterns, in the short and in the long terms, in the time series of invention specific to how knowledge accumulates and builds on itself through a process of collective learning. Detecting the traces of collective learning using the history of the patent record is not analytically trivial since the patent system is embedded in a larger socioeconomic system. Because of this, we study only aggregate patterns over the longest time frames available in order to minimize the statistical impact of second order effects. Notwithstanding these difficulties, we present evidence that patents indeed accumulate according to a process of collective learning, a finding consistent with the idea that invention is a process of cultural accumulation (12).

We emphasize that the mere detection of accumulation in the historical record of invention is not, by itself, a sufficient indicator that an underlying process of learning must have occurred. In fact, many important processes of accumulation in socioeconomic systems are not processes of cultural accumulation. An example is population growth. The "accumulation" of living people on the planet is still not fully understood, but is presumably a consequence of a deepening of the carrying capacity of our social systems (13). Another example is the accumulation of $CO_2$ in the atmosphere, caused by human activity. It is likely that both these examples are *con-*

*sequences* of cumulative culture (14), but they are not *themselves* processes of cultural accumulation. The accumulation of patents is interesting because it may be a very specific type of accumulation. Our task is to investigate whether we can identify invention itself as a process of cultural accumulation.

We report three main main findings. First, we find that the microstructure of stochastic fluctuations in the time series of cumulative patents displays the presence of memory, in the sense that changes in the inventive output have an enduring multiplicative effect on all subsequent rates of patent growth. Second, we find a long-term transition from sub-exponential to super-exponential growth, which is surprising, since models of innovation only predict exponential or super-exponential growth. Sub-exponential growth could be caused by the increasing difficulty and complexity of patenting (15). Such explanation is at odds, however, with the recent surge in patenting (16). And third, we present a simple model that is able to accommodate all these observations, reconciling this transition in the historical patterns of inventive activity with the idea that invention is a process of cultural accumulation in which old ideas recombine with recent ideas to generate new ones.

We believe our results are interesting for at least three reasons. Firstly, they bolster the view that the patent system is yet another social practice that humans have devised to coordinate their actions in order to act as a collective brain (12). Secondly, the results presented here imply that the U.S. patent system does indeed accomplish the mission codified in article 112 of the U.S. Patent Code which states that "patents must clearly disclose and *teach*" (italics added for emphasis). Thirdly, as becomes clear from the analysis below, we conclude that a general process of collective learning cannot be identified, in general, through a single kind of growth (e.g., sub-exponential, exponential, or super-exponential) as has been documented elsewhere (17), but can instead display a combination of growth regimes.

# 2 Invention and Learning

Processes of continuous and open-ended growth seem to be a unique feature of the human species. No other species has grown in population and ecological extension so much as ours. Another distinctive feature of the development of *Homo Sapiens Sapiens* has been the scale and diversity of its social organization. Why has this growth not saturated or even halted? Given the remarkable differences in size and heterogeneity of human social systems as compared to other animals, a central question in the field of human evolutionary biology has been why do social systems grow in size and diversity at all (18)? The answer seems to be humans' capacity for *culture*.

Work in the budding field of cultural evolution suggests that a lot of the growth exhibited by socioeconomic systems is fueled by a process of accumulation in which the solutions to problems found in a generation are handed down to the next generation (19; 14). The claim is that extensive growth in social systems, like growth in population size, is the consequence of accumulating a large number of tools, norms, beliefs and knowhow that make a social system adaptive, and make it grow as it can occupy a greater range of environmental niches (19). The accurate transmission of information from one generation to the next allows solutions to complex problems to accumulate. Humans have been able to create a tool-kit (containing physical and intellectual tools) allowing human groups and populations to solve problems that are much too hard for individuals to solve by themselves. However, the transmission has to be accurate enough so that small innovations are collectively remembered more often than they are forgotten.

Collective learning was as important for the development of new knowledge and technologies for hunter/gatherers hundreds of thousands of years ago as it is was for the development of

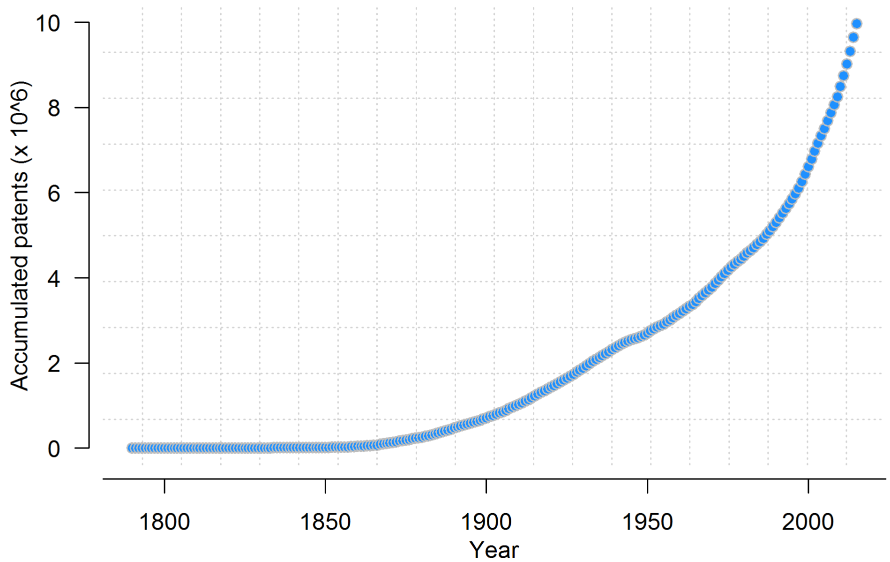


Figure 1: All utility patents granted by the U.S. Patent Office by application year. (A "utility patent" is a patent that covers the creation of a new or improved, and useful, product, process or machine. Utility patents make up more than 90% of all granted patents.)

modern science and the practice of contemporary invention (8). If, to paraphrase (20), the central fact about the processes of invention is that it is devoted to the production of information, the process is driven by the sharing of information. Recent historical analyses highlight the importance of information sharing, and of the social, economic and cultural preconditions that facilitated such sharing, in the genesis of "modern science" (21; 22; 23). Scholars of innovation from the time of the Industrial Revolution to the present have emphasized the importance of knowledge sharing as a fundamental source of inventions. This sharing was important because it not only diffused new techniques, but it also facilitated further, cumulative improvements (5; 24; 25; 26). Knowledge sharing is not mainly a modern development, as revealed by an examination of the history of invention before the 18th century (27). Nevertheless, there are indications that knowledge sharing—as signified by the prevalence of knowledge networks linking Universities and firms and the dominance of teams in the production of knowledge—is a salient characteristic of contemporary invention (28; 29; 30; 31; 15; 32).

Does Figure 1, which shows the entire time-series, spanning more than two centuries, of the total number of utility patents in the U.S. patent system, portray a process of cultural accumulation? Or is it just the time series of a process of accumulation of items in a historical record? One indication that invention is a process akin to cumulative culture is the evidence that patenting is a combinatorial process (33). Each patent represents a novel combination of ideas. Ideas are coded as "technologies" in the U.S. patent system, and together with the set of 'claims', they define what a patent is. Technologies can be regarded as the atomic basis of patents and can themselves be new or old. That is, a patent can simply use extant technologies, or introduce new ones.[1] Patented inventions are

[1] The definition of "technology" used by the U.S. Patent Office is more than adequate for our purposes: the "application of science and engineering to the devel-

increasingly the result of combinations of extant technologies (33).

The discreteness inherent to the inventive processes has been recognized by many scholars before and has been deemed as essential to the process of invention (36; 37; 38; 39; 40). From the vantage point of the field of cultural evolution, we ask again the question posed by Martin L. Weitzman in reference (38): "What are the generic mathematical properties of a combinatoric growth process?" At first sight, two generic properties appear to emerge from the combinatorial nature of invention: super-exponential growth (that is, growth that is faster than exponential) and path-dependency (41; 38; 39). To what extent are these properties present in Figure 1?

More than half a century ago Harvey C. Lehman provided the first thorough review of the accumulation of culture in different areas of knowledge such as philosophy, medicine, economics, and many others (using data that started in 1200AD for some areas) (42). He concluded that accumulation in all these fields follows an exponential curve. Enquist et al. (2008) (43) posits that culture begets culture, and this explains why the accumulation of culture induces exponential growth. But how to reconcile the super-exponential growth predicted by models of recombination of ideas with the observed exponential growth of culture? Is the exponential growth an indication that the super-exponential production of culture is constrained by the limitations of society to absorb new ideas as argued by (39), and if so, is culture limited to grow according to the growth of population size? Or should we consider the accumulation of culture a different phenomenon from the recent phenomena of technological progress, given that some series related to distinctly modern knowledge-based activities do indeed exhibit super-exponential growth, such as performance curves in aerospace propulsion systems (44), computing performance (45) or returns in financial option markets (46)?

Enquist et al. (2011) (17) show that cultural growth can alternate between linear, exponential, or super-exponential depending on how cultural traits are acquired. For example, if cultural traits are acquired independently from one another, culture will grow linearly. This process, then, should not be referred to as cultural accumulation since there is no sense in which a population of individuals interact or coordinate their actions in the process. (17) show that if cultural traits are not acquired independently but instead depend, somehow, on each other, culture can grow following different functional forms. In particular, they show that exponential growth occurs when accumulation is driven by differentiation, i.e., when an element becomes two slightly different new versions (e.g., branching), and that super-exponential growth occurs when accumulation is driven by recombination. This provides a clear distinction between processes of *collective learning* from collective processes of *individual learning*. According to (17), the former produce exponential or super-exponential growth, while the latter will only produce linear growth. We examine next which of these patterns, if any, are present in Figure 1.

Path dependence is much harder to conclusively identify in the historical record. Partly because it is an "umbrella" term to refer to many different phenomena (see (47) for a review and clarification of all the facets behind the notion of path dependence). Paradoxically, however, it is perhaps one of the topics more extensively discussed in the literature on technological change (48; 49; 50). Some scholars have recently proposed new mathematical models that incorporate the serendipitous aspects of invention, how these can open or close further paths of invention and sometimes lead to waves of innovation (51; 52; 53). With these models one can derive statistical laws describing nov-

opment of machines and procedures in order to enhance or improve human conditions, or at least to improve human efficiency in some respect" (34) citing (35, p.384).

elties, such as their temporal-burstiness (time clustering), their growth (e.g., Heaps' law) and probability distribution (e.g., Zipf's law). Our study of path dependence in the patent record differs from these previous works in a simple aspect. We will ask whether "shocks" in the rate of invention have lasting effects in subsequent production patents. We will frame evidence of this by showing that the time series of invention display "memory" in a statistical sense.

In the next sections we describe the data and the methods used to examine the accumulation of patented inventions as an instance of cultural accumulation driven by collective learning. We will first study the evidence for memory in the time series of patents. We will follow with an analysis of the long-term growth of patents to look for exponential or super-exponential patterns. We then present a simple mathematical model that reproduces some of the empirical findings.

# 3 Materials and Methods

## 3.1 Unit of invention

Anthropologists study culture by identifying cultural "traits". The "cumulative" part of culture stems from the accumulation of traits through generations. But what is a cultural trait? The claim that culture changes through a process akin to Darwinian evolution has as a consequence that culture can change by drift, or by adapting to social or physical environmental challenges (54; 55). The unit of replication (i.e., a cultural trait) must take the form of "mental representations" that humans can learn and teach easily, which, by their Darwinian origin, solve specific challenges or accomplish particular tasks (56). If we want to compare the process of invention to that of the accumulation of culture, what should be used as the equivalent of a cultural trait?

The notion of "unity of invention" in patent law has the purpose of restricting patents to only those inventions that solve a specific problem or accomplish a particular task. One of its purposes is to preclude bundling several inventions in a single patent application. Here, we appeal to the unity of invention to argue that the patent record is a system of accumulated solutions whose size can be quantified simply by counting patents in the system. Since cumulative culture is the social phenomenon of accumulating solutions to problems, the unity of invention principle allows us to use patents as our unit of analysis. We count patents by their year of application, as is custom, to best account for the moment an inventor considered he or she had a new solution to a problem good enough to apply for a right to exclusion.

## 3.2 Quantifying the presence of memory

If invention is to be understood as a process of cultural accumulation, then patents are at the same time the input and the output of a process of collective learning. Thus, we want to emphasize that patents are "tools for learning", and as such, a patent can (and should) in principle affect permanently the rate of future learning (and by implication the future rate of invention).

The hypothesis we want to test is whether a stochastic shock in the rate of invention has an effect on the future rate of inventions that is permanent or transitory. This hypothesis can be tested through a *unit root test*. The simplest stochastic process with a "unit root" is

$$x(t) = ax(t-1) + \varepsilon_t, \qquad (1)$$

when $a = 1$. The term $\varepsilon_t$ is a stationary, serially independent, mean zero noise random variable. The essence of a sequence $x(t)$ with a unit root is that it depends on what has happened before. The value of $a$ in Equation (1) determines how much the past affects the future. In the simple case of Equation (1), replacing iteratively, we get that

$$x(t) = a^t x(0) + a^{t-1}\varepsilon_1 + \ldots + a\varepsilon_{t-1} + \varepsilon_t.$$

If $a < 1$, the effect of past shocks $\varepsilon_t$ on the future decay exponentially, and $x(t)$ has a stationary behavior. If $a > 1$, then past shocks are increasingly amplified with time and $x(t)$ has some "explosive" and unstable dynamics. But if $a = 1$, then all past events affect the present equally, and $x(t)$ is said to have memory, or technically speaking, to have a 'unit root'.

Typical time series contain both a stochastic and a deterministic component. Thus, it must be emphasized that in this section we are interested in the behavior of the stochastic component in our time series (in the next section we will address the deterministic component). As is custom, we use the *augmented* Dickey-Fuller (ADF) test as the unit root test, which allows a more flexible time-series model than the usual Dickey-Fuller (DF) test for Equation (1). To complement the ADF test, we use the Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test to assess trend stationarity. We will sequentially take time differences of our time series, seeking the point in which (i) we *cannot* reject stationarity (KPSS test) *and* (ii) we *can* reject the presence of a unit root (ADF test). If both conditions (i) and (ii) are met at a given stage of differencing, then there is evidence that the time series *before* the last differencing has a unit root. We interpret evidence of a unit root as evidence of memory. We leave some additional details about testing unit roots in Appendix A.

### 3.3 Characterizing long-term growth

Exponential growth is usually conceived as "disruptive" or "interesting" (e.g., 57). Mathematically, however, exponential growth is quite simple. It occurs when a variable doubles whenever a fixed amount of time has elapsed. What is interesting is the fact that exponential growth is at the boundary between two very different patterns of growth. We will say that the growth is sub-exponential when the amount of time needed for the variable to double becomes longer and longer as the variable grows, and we will say it is super-exponential when the doubling time becomes shorter and shorter. A special property of the latter is the existence of a moment in time in which the variable becomes infinite. Since this cannot happen in reality, this so-called "finite-time-singularity" typically represents a phase-shift.

As we discussed in Section 2, these different types of growth can be used to characterize invention, and use this characterization to learn whether invention is a process of collective learning based on the theories that have been proposed elsewhere.

These three different types of growth can be characterized by a single parameter $\delta$ through the following differential equation

$$\dot{x}(t) = \gamma x(t)^{\delta}, \tag{2}$$

where $x(t)$ is the accumulated number of patents until year $t$, $\dot{x}(t)$ is its time-derivative (i.e., the new patents in a relatively small interval d$t$, possibly a year), and $\gamma$ and $\delta$ are coefficients on the real line. Solving Equation (2) to get $x$ as a function of time,

$$\int_{x_0}^{x(t)} \frac{\mathrm{d}x}{x^{\delta}} = \int_{t_0}^{t} \gamma \mathrm{d}t, \tag{3}$$

yields the general solution:

$$x(t) = x_0 \left[1 + \left(\frac{\gamma - \gamma\delta}{x_0^{1-\delta}}\right)(t - t_0)\right]^{\frac{1}{1-\delta}}. \tag{4}$$

Depending on the value of $\delta$, Equation (4) models our three types of dynamics we are interested in:

$$x(t) = \begin{cases} c_1 \left((x_0/c_1)^{1-\delta} + (t - t_0)\right)^{\frac{1}{1-\delta}} & \text{for } \delta < 1 \text{ and } t > \\ x_0 \; \mathrm{e}^{\gamma(t-t_0)} & \text{for } \delta = 1 \text{ and } t > \\ \dfrac{c_2}{(t_{\text{critical}} - t)^{\frac{1}{\delta-1}}} & \text{for } \delta > 1 \text{ and } t_{\text{crit}} \end{cases}$$

where $c_1 = (\gamma(1-\delta))^{1/(1-\delta)}$, $c_2 = (\gamma(\delta - 1))^{-1/(1-\delta)}$, and $t_{\text{critical}} = t_0 + \gamma(\delta-1)x_0^{\delta-1}$ is the critical time that defines the finite-time singularity when $\delta > 1$.

Equation (5a) describes sub-exponential growth, Equation (5b) exponential growth, and Equation (5c) super-exponential growth. Empirically, we will test which of these curves fits the data best by estimating the parameter $\delta$ on a linear regression using Equation (2).

# 4 Results

## 4.1 Memory in invention

Figure 2 shows, on the left panel, the accumulated patents $x(t)$ and its first and second differences, and on the right panel, the logarithm of $x(t)$, and its first and second differences. Note that the first difference of $\ln(x(t))$ is approximately the percentage change, $r(t)$, of $x(t)$.

Our main finding here is the plot shown at the bottom-right of Figure 2. This plot shows the point where we can reject the hypothesis of a unit root and simultaneously cannot reject the hypothesis that this is a stationary series. In other words, both tests suggest strong evidence of a stationary sequence of random noise describing the change in growth rates from year to year.

In this bottom-right panel of Figure 2, around the 1850's, we observe an initial period in patenting in which the series of $\Delta r(t)$ has an unusually large variance. After this period, however, the time series of the yearly change of the rate of growth of the accumulated patents is more regular. That is, $\Delta r(t) \approx \mu(t) + \varepsilon_t$, where $\varepsilon_t$ are serially uncorrelated variables and $\mu(t)$ is possibly a deterministic trend, although most probably equal to zero in this case.

The results of the statistical tests imply that the series just above the bottom-right plot in Figure 2 has a unit root. We can see this by noting that $\Delta r(t) \approx \mu(t) + \varepsilon_t$ implies that the rate of change of patenting is the same as rate of change of the previous period, plus something that can be deterministic with some stationary noise, $r(t) = \mu(t) + r(t-1) + \varepsilon_t$. Hence, we say that growth rates $r(t)$, and in turn the accumulated patents $\ln(x(t))$, have memory. The results of the statistical tests, however, do not tell us much about the deterministic components of these series, which characterize the long-term growth patterns of invention.

## 4.2 The long-term trends of invention

The last section presented evidence that the stochastic part of the patenting time series has memory. We now turn our attention to the long-term trend (the "deterministic" part in the language of the last section). We want to know if the time series of invention is consistent with previous models that predict either an exponential or a super-exponential growth process for the learning-mediated accumulation of knowledge.

We take logarithms on both sides of Equation (2) to carry a linear regression and study the value of the coefficient $\delta$:

$$\ln(y(t+1)) = \ln(\gamma) + \delta \ln(x(t)) + \epsilon_t, \qquad (6)$$

where $y(t+1) = x(t+1) - x(t)$ is the first difference of $x(t)$.

Figure 3 shows this correlation, and we find that $\widehat{\delta} \approx 0.772$. The estimated value of this coefficient implies that a 1% increase in the total body of patents is historically associated with an average increase of 0.772% in the number of new patents that are produced in the subsequent year. The result of this simple regression shows that, overall, the accumulation of patents has been sub-exponential over the whole period of time. Thus, the general trend is better described by the first solution given in Equation (5a). Sub-exponential growth is unexpected because none of the models we reviewed predicted such trend. In addition, it is unexpected given that the population of inventors does not grow sub-exponentially. And yet, it is a systematic pattern over approximately 200 years. The sub-exponentiality of invention can be observed already in Figure 2, as the concavity of top-right's plot.

The long-term trend in Figure 3, however, reveals more than simple sub-exponential growth. One can also observe that there are some subtrends and some sustained deviations from the line.

Deviations from pure sub-exponential growth are observed in more detail in Figure 4. Patents grow slower, equal or faster than exponentially on different periods. We quantify this dependence of parameter $\delta$ on time by running the same regression as in Equation (6), but for limited windows of 50 years. We let this 50-year window roll over the whole sample. Figure 4A plots the estimated value $\widehat{\delta}$ for each window. The error bars represent $+/-$ one standard error. Broadly, there are four epochs: from 1789 to 1836, from 1837 to 1868, from 1869 to 1944, and from 1945 until 2015. Breaking the time series at those years, we observe that there are small periods of super-exponential growth. In Figure 4B we show the estimated regression lines of those four periods. Red points move away from the $45^o$ dashed line, hence they represent sub-exponential growth; Blue dots move towards the $45^o$ dashed line, hence they represent super-exponential growth. We find $\widehat{\delta}_{1789-1836} = 0.56$ ($\pm 0.04$), $\widehat{\delta}_{1837-1868} = 1.64$ ($\pm 0.07$), $\widehat{\delta}_{1869-1944} = 0.46$ ($\pm 0.02$), and $\widehat{\delta}_{1945-2015} = 1.58$ ($\pm 0.06$). We plot in 4C the accumulation across years and these breaks.

It is interesting to note that the period after 1869 is recognized by historians as a moment of rapid economic growth in the US where the "American system of invention" played a central role (58). And yet, the period from 1869 until the Second World War is the longest period in which patents accumulated sub-exponentially. Only since after the War that patents have been accumulating super-exponentially. In the next section we develop a model that reproduces the observed patterns and is consistent with the claim that the accumulation of patents is a process of cultural accumulation. In this way, we show that a single process can explain the transition from sub- to super-exponential growth.

## 4.3 Mathematical model

Three main drivers are typically considered to explaining the production of inventions (59; 12): the size of the population of inventors, interactions between inventors (i.e., collaborations and spillovers), and the recombination of inventions. These three drivers correspond to facilitators of invention in human societies (assuming that useful inventions are rare): larger populations (*i.e.*, more brains) have a greater likelihood of generating inventions (60; 18; 61); a connected population of inventors increases the likelihood that information flows will result in increased invention rates (62); and invention is predominantly a process of combining existing technologies to generate novelty (e.g. 63).

In order to offer an analytic explanation of the observed patterns of patenting growth, we first examine what is the relation with population size. Given that the majority of patents in the USPTO are originated in the U.S., we plot in Figure 5 the number patents produced per decade and the U.S. population according to the census at the origin of the decade. On the left panel we can see that the production of patents is more variable in time than population, while on right panel we see the association between the two variables. The two variables are associated through a power-law relationship, but we observe some sustained deviations.

The model we present in the next section excludes population size and the interactions between inventors. We focus instead on the idea of combinatorial growth in order to explain the transition from sub-exponential to super-exponential growth. Models treating population size and interactions explicitly are discussed briefly in Appendix B.

### 4.3.1 Simple model of combinatorial growth

The model we propose here is based on Weitzman's model of recombinant growth (39). Weitzman's null model states that the new patented inventions added to the system are the result of combining all the possible pairings between patents that have not been crossed before:

$$\overbrace{x(t+1)}^{\text{next}} = \overbrace{x(t)}^{\text{current}} + \alpha \left( \underbrace{\binom{x(t)}{2}}_{\text{all current combinations}} - \overbrace{\binom{x(t-1)}{2}}^{\text{all previous combinations}} \right), \tag{7}$$

where $\alpha$ is a constant and $\binom{n}{k} = n!/(k!(n-k)!)$ is the number of different ways one can choose $k$ elements among $n$. Note that only pairwise combinations are considered in this model, although Weitzman generalized his results to higher order combinations.

Weitzman's work is focused on showing that the process of growth in Equation (7) is super-exponential, which means it will reach a finite-time singularity and, consequently, is not sustainable. In other words, the process must eventually encounter some constraint. Weitzman assumes the constraint is the population's capacity to process the overproduction of all new ideas (39). We will not dispute this conclusion. Instead, we formulate a modified model whose goal is to also predict super-exponential growth, but only after a phase of sub-exponential growth, as observed in the historical record of patents.

The modified model is very similar to Equation (7), but with a subtle change. The difference between Equation (7) and the model we are about to present is that in Weitzman's model patents are allowed to recombine with other cotemporaneous patents, at the moment they are generated. In our model, at least one period has to pass, and then only the most recently added patents will be able to combine with (a fraction of) the previously existing ones. This restriction is consistent with the fact that the prior art of new patents overwhelmingly refers to recent patents (64). This minor modification has significant consequences.

The model's building blocks are:

1. A fraction $p_{\text{old}}$ of old patents at any given point in time are recombinable. Thus, a fraction $1 - p_{\text{old}}$ of the old patents are forgotten or *obsolete* and hence cannot be used for recombinations;

2. There are some constraints from the *specificity* (or specialization) of patents that prevents the new patents from combining with the old patents, and thus only a proportion $p_{\text{new}}$ of the most recently added patents (i.e., added in the most recent period) can be combined with the old ones;

3. And finally, we assume not all possible recombinations are in fact *discovered* but only a fraction, $p_{\text{disc}}$.

Putting all these assumptions together yields

$$x(t+1) = x(t) + p_{\text{disc}}[p_{\text{new}}(x(t) - x(t-1))][p_{\text{old}}x(t-1)]. \tag{8}$$

The constants can all be combined such that $\alpha = p_{\text{old}}p_{\text{new}}p_{\text{disc}}$. The model becomes:

$$\overbrace{x(t+1)}^{\text{next}} = \overbrace{x(t)}^{\text{current}} + \alpha \underbrace{\underbrace{x(t-1)}_{\text{old patents}} * \underbrace{(x(t) - x(t-1))}_{\text{most recently added patents}}}_{\text{recombination}}. \tag{9}$$

There is no analytic solution for Equation (9). However, Equation (9) can be re-expressed in continuous time which can lead to a more tractable differential equation (we show the mathematical steps for doing so in Appendix C). In continuous time, we write the model as

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = cy(t)(x(t) - 1), \tag{10}$$

where $c$ is a new constant replacing the role of $p$. The solution is

$$x(t) = 1 + \frac{K}{c} \tan\left(\frac{K}{2}t + \tan^{-1}\left(\frac{x_0 - 1}{K/c}\right)\right), \tag{11}$$

where $K = \sqrt{2y_0c - c^2}$ and $x_0 = x(0)$.

The trigonometric *tangent* function has interesting properties and predicts an interesting behavior for the growth of patents. It has an initial sub-exponential growth but then gradually transitions into super-exponential growth. Thus, tangent growth leads to a divergence at a "critical time"

$$t_{\text{critical}} = \frac{\pi}{K} - \frac{2}{K}\tan^{-1}\left(\frac{x_0 - 1}{K/c}\right).$$

One can re-express Equation (11) as

$$x(t) = 1 + \frac{K}{c}\tan\left(\frac{\pi}{2} - \frac{K}{2}(t_{\text{critical}} - t)\right). \tag{12}$$

For the empirical validation of such simple model, we fit the following equation:

$$x(t) = 1 + A\tan\left(Bt + \phi\right), \tag{13}$$

where $A$, $B$, and $\phi$ are free parameters we need to estimate. We compare the fit of Equation (13) to that of an alternative model with cubic terms, with also three parameters:

$$x(t) = A + B\left(t - C\right)^3. \tag{14}$$

We refer to this model as "Cubic Model" in Figure 6.

The estimation of these models is done to the data post-1870, since that date is after the short period of super-exponential growth. Figure 6 shows the fits of both models, putting the $y$-axis in logarithmic scales. According the estimated values of the parameters of Equation (13) we can recreate the parameters of Equation (12), in particular the critical time. For that we get the year $\widehat{t}_{\text{critical}} \approx 2051$.

The model presented in the previous section is incomplete: it treats the process of invention as an abstract process of recombination and does not take into consideration the endogenous relationships between technological progress, economic growth, and population size. While there is ample evidence that larger populations have higher inventive activity (31), it remains as future work to model explicitly the cross-sectional patterns connecting invention to demographic variables together with the longitudinal patterns found here.

# 5 Discussion

The motivation for the present work was to assess the view that the U.S. patenting system is an institution that promotes collective learning akin to the process of cultural accumulation, and learn whether the historical series of patenting present clear empirical "signatures" for such a process. We used the framework of cultural evolution to generate two hypotheses: first, the time series of patent accumulation ought to have memory; second, the overall pattern of growth should be consistent with a model of recombination of ideas. The first hypothesis is about the characteristics of the fluctuations on short time scales, while the second hypothesis is about the trends on the long term. In our empirical investigation, we used the series of total number of patents granted in the U.S. by their application year, from 1790 to 2016, and we found evidence supporting both hypotheses. These results underline the connection between patenting, combinatorial growth, and the process of learning by accumulating knowledge.

Specifically, we found that the small fluctuations in patenting rates have a "unit root". This result suggests the system has memory, whereby shocks in the patenting rate shift permanently future rates. Thus, patents are not only inventions which solve past problems, but tools that teach future generations of inventors how to solve future problems. In this way, a set of inventions patented in a given year can permanently change the rate of future patenting.

With regards to the long-term accumulation of patents, we found that new patents produced in a given year to be a power-law function of the total number of accumulated patents until that moment. Power-law relationships like these are typical in the literature of "learning curves" (e.g., see (65)), and suggest a systematic process of cumulative learning. Thus, the growth of patenting is more tightly connected with the size of patents produced thus far, than with how much time has passed since the process started. Put differently, it is the body of accumulated knowledge itself which drives the production of new inventions. It is consistent with anthropologists' view of a process of cumulative culture. Notably, the value of the exponent in this power-law implied that the series is not characterized by neither exponential, nor super-exponential, growth, as predicted by conventional models of innovation. Instead, we found a more involved pattern in which the accumulation is characterized by a first phase of sub-exponential growth followed by a super-exponential phase.

Notably, over the span of the U.S. Patent Office history, we found that the cumulative number of patents has grown sub-exponentially for the most part. Such growth is unexpected given the approximately exponential growth of population, increased channels of information flows, and the accelerated process of globalization since the Industrial Revolution. The connection between demography and invention is a question that has received a lot of attention in anthropology and economics (see, for example, (66; 67; 60; 18; 59; 68; 69)). The patenting record suggests that although there is a relationship between invention and population size it is not a straightforward relationship, and inventions have dynamics that play on top of those of population growth. For example, despite the fact that fertility rates have been falling steadily (especially in high-income countries), we found that patent growth has become super-exponential since the Second World War.

We proposed a simple combinatorial model that reconciles all these observations by assuming that patents can only recombine to produce new patents (33). Crucially, however, patents in our model cannot recombine with contemporaneous patents, and this allows the model to accommodate an initial phase of sub-exponential growth followed by a transition to super-exponential growth. This type of growth, which is well described by a tangent function. Future work should aim to find this same empirical signature in other systems of collective learning.

We do not claim our model to be a realistic representation of the process of invention, but it serves as evidence that aspects that are deemed as essential in the process of invention—recombination and building on the past—can reproduce these broad empirical observations.

The word "patent" comes from the Latin word "patere", which means "to lay open". Historically patents have had the mission of disclosing new practical knowledge clearly enough that others can use it and build upon it (70; 71). Patenting systems can be therefore presumed to be institutions that coordinate and promote the accumulation of know-how, with the patent record providing evidence of such an accumulation. If this presumption is valid, then patenting should grow like culture grows, and the processes of invention should mimic those of cultural accumulation. The results presented here are a modest contribution towards a new perspective on patenting, namely as an instance of cultural accumulation.

If our findings keep having empirical support in the future, then they may change how we conceive the process of invention. In particular, they will lend support not only to the view that invention is a collaborative effort, but also that it is truly a process of collective learning across generations.

Author contributions: A.G.L. performed

research, wrote programming and statistical code. D.S. compiled and curated data. A.G.L. and J.L. designed research. A.G.L., J.L., D.S., and J.H. wrote the paper, and gave final approval for publication.

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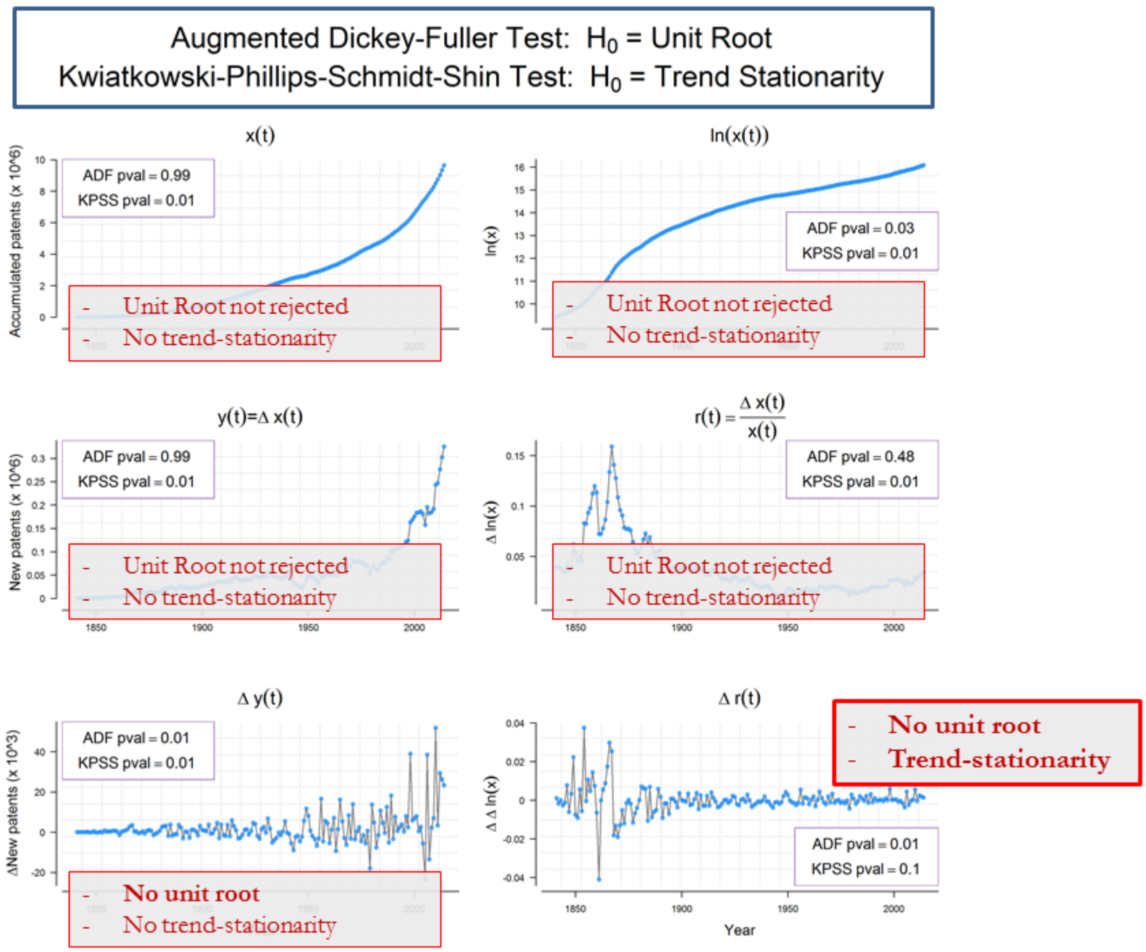


Figure 2: Testing unit-root and trend stationarity. In all panels, the $x$-axis represents time in years. The plots in the left column are analysis of the raw accumulated patents per year, $x(t)$ (top), its first difference $y(t) = \Delta x(t)$ (middle), and second difference $\Delta y(t)$ (bottom). The plots on the right are analysis of the logarithm of accumulated patents per year, $\ln(x(t))$ (top), its first difference $r(t) = \Delta \ln(x(t)) \approx \Delta x(t)/x(t)$ (middle), and second difference $\Delta r(t)$ (bottom).

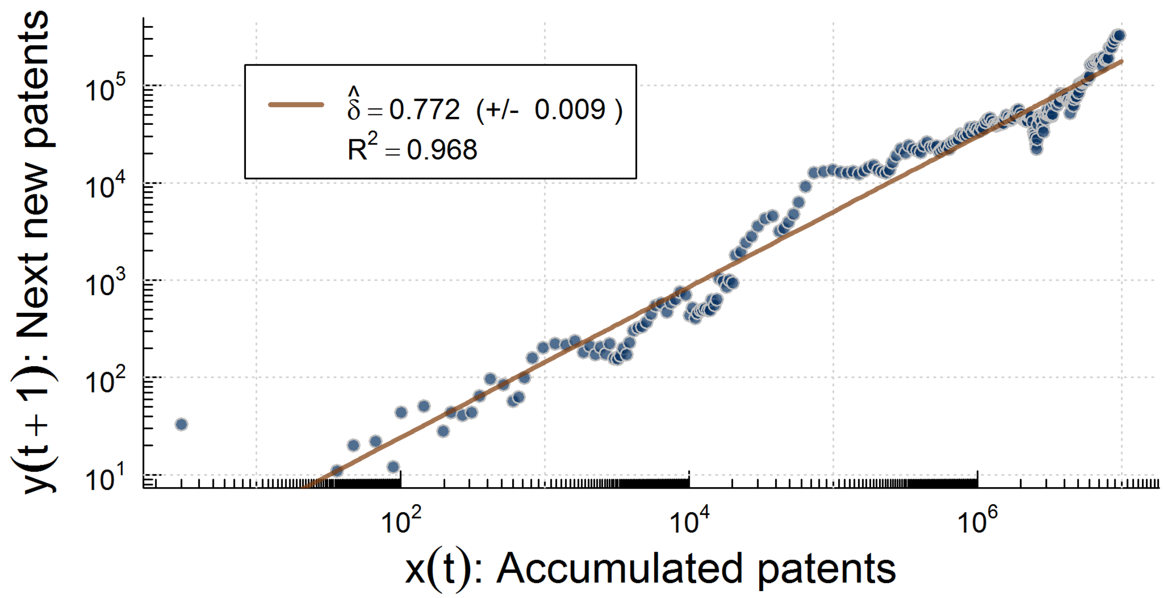


Figure 3: New patents each year as a function of previous year's accumulated patents.

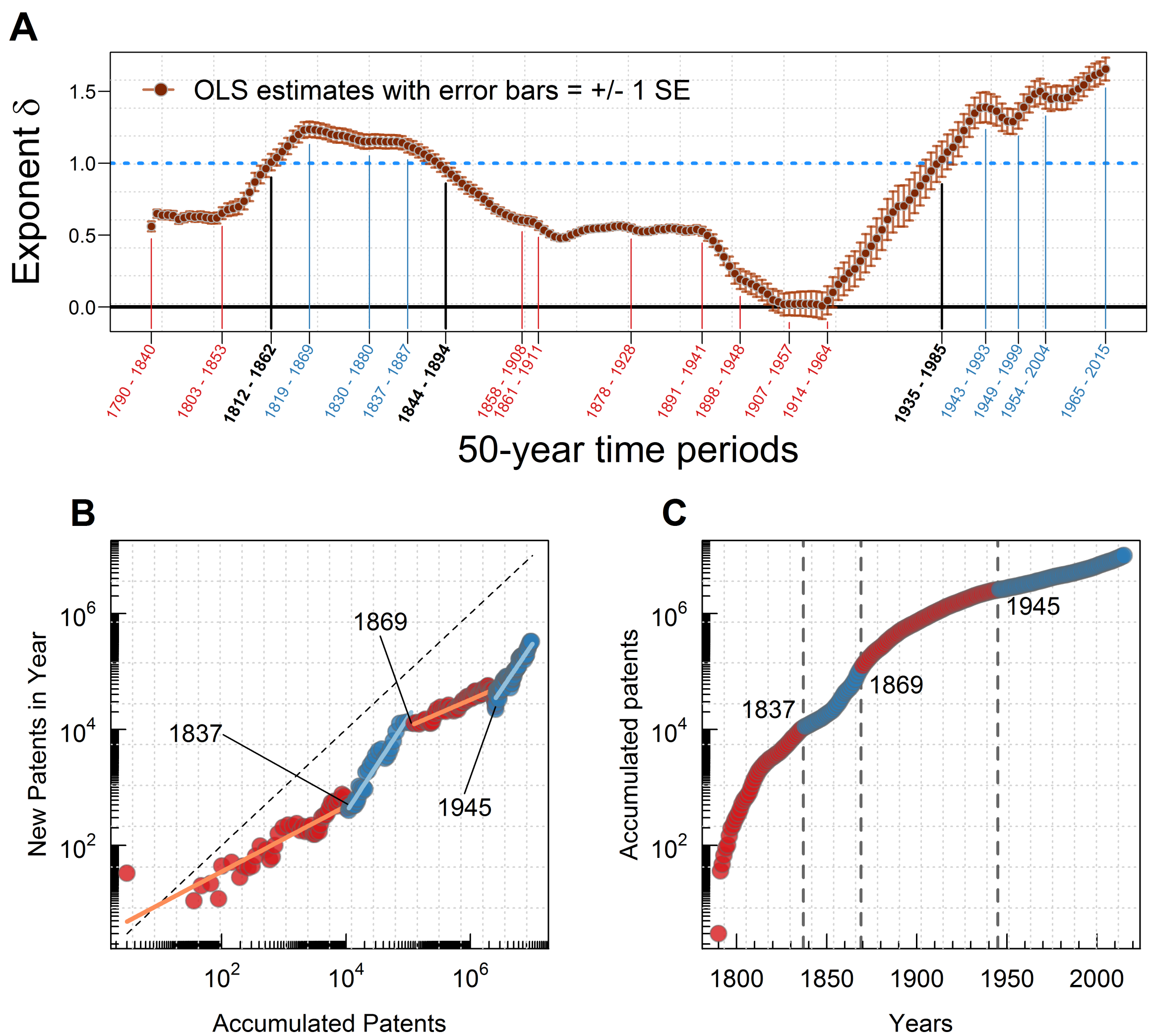


Figure 4: **Top panel:** The graph shows the estimation of this exponent over many sequential running windows of 50 years, from 1790 until 2015. It is observed that the value changes in different periods of time. The colored vertical lines identify sudden changes in the behavior of the exponent over time, where red are those periods where growth was sub-exponential and blue super-exponential. The black vertical lines identify the periods in which there was a change in regime from sub- to super-exponential growth, or vice versa. **Bottom panels:** The plot on the left shows the scatter of the new patents as a function of the accumulated patents in log-log scales, and the plot on the right shows the accumulated patents as a function of time, with the y-axis shown in a log scale. In both plots, red dots represent the periods of time in which patents accumulated sub-exponentially, while blue dots represent periods of super-exponential growth.

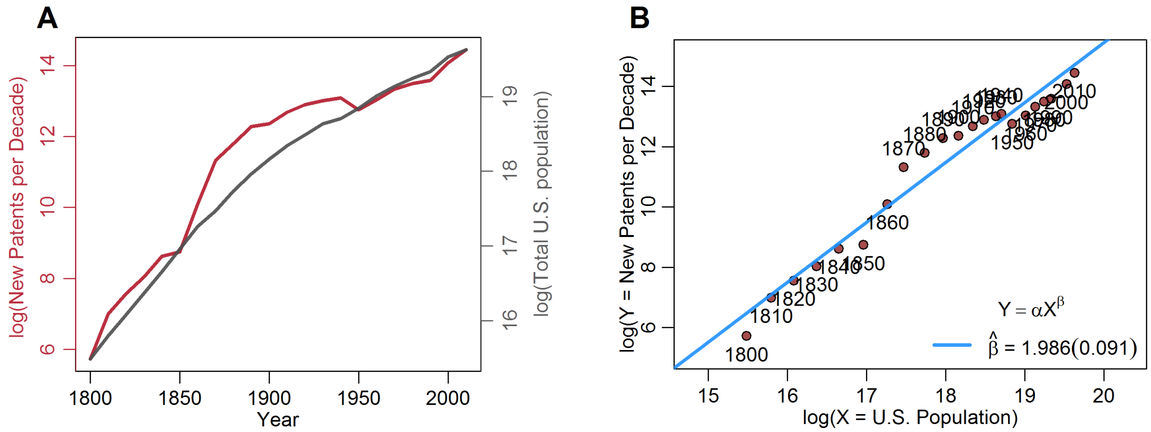


Figure 5: Longitudinal scaling analysis of total U.S. population size per decade and its association with the production of patents during the subsequent decade, from 1800 to 2010. Panel **A**: Time series of total patents produced in each decade (left axis, red line) and the U.S. total population census at the origin of the decade (right axis, gray line). Panel **B**: log-log association between patents produced per decade ($y$-axis) and U.S. population ($x$-axis), following an approximate power-law $Y = \alpha X^{\beta}$.

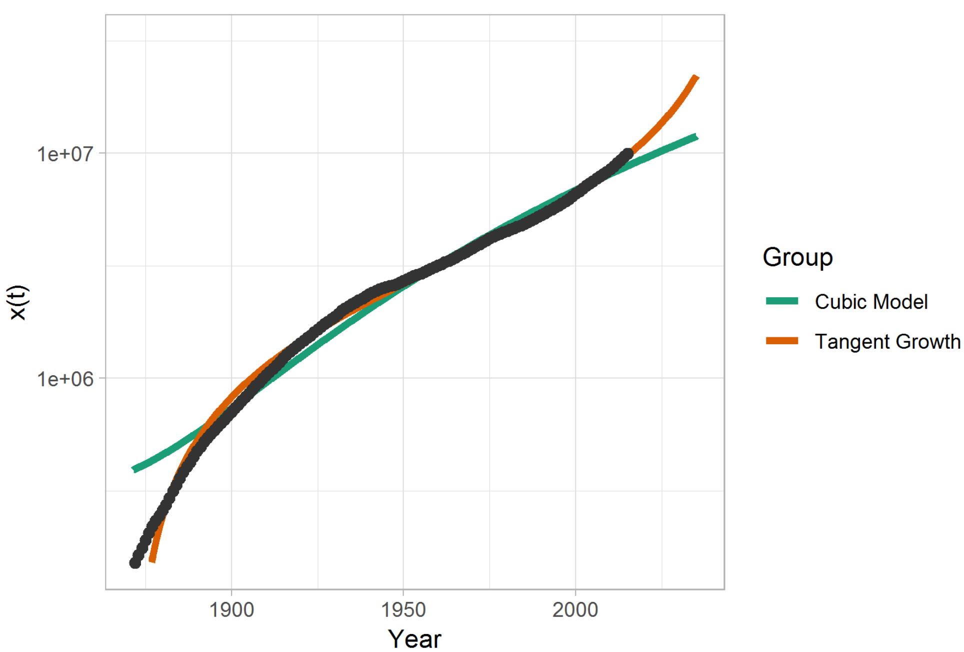


Figure 6: Lines are fits to the data (black dots). Both fitted models have three parameters.

# Appendix A. Testing unit roots

A naive test for the presence of a unit root, one performs a regression of $\Delta x(t) \equiv x(t) - x(t-1)$ against $\beta_0 + \beta_1 x(t-1)$. If there is a unit root, then $\beta_1 = 0$, and $\widehat{\beta_1}$ should not be significantly different than zero.

In practice, it is hard to reject the hypothesis of the unit root if one simply tests for $H_0 : a = 1$. Time series often pass this unit root test simply because time series are typically neither stationary nor 'explosive' (see discussion by 72). Thus, one seeks in practice to find the stage at which the null hypothesis of the unit root can be rejected by differencing the series several times (i.e., $y(t) = \Delta x(t)$, $z(t) = \Delta y(t)$, etc.). At that point, the time series should be stationary. This implies that one should also aim to test for stationarity *in addition* to unit root. By "stationarity" we mean "trend stationarity", which refers to a time series like $x(t) = \mu(t) + \varepsilon_t$, in which $\mu(t)$ is a deterministic trend while $\varepsilon_t$ is stationary noise.

The *augmented* Dickey-Fuller (ADF) test allows a more flexible time-series model than the usual DF test. For trend stationarity, we use the Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test. As we sequentially take time differences of our time series, we are looking for the point in which (i) we *cannot* reject the KPSS test *and* (ii) we *can* reject the ADF test. If both conditions (i) and (ii) are met at a given stage of differencing, then there is evidence that the time series *before* the last differencing has a unit root. Evidence of a unit root is therefore evidence of memory.

We note that a unit root test can be utilized to test whether the growth of a system has experienced a regime change (73) (see, however, (72)). (74) deploys a unit root test to assess whether the rapid growth in patenting noticeable in the United States starting in the mid-1980s truly represented a break from past growth dynamics. Here we are interested in the characterization of the growth of patents over a longer period and how such growth constraints any explanatory account of the underlying processes driving the observed growth.

# Appendix B. Model of patents as the output of inventors interacting in cities

To capture the effects of scale and connectivity on invention we proceed by treating (as the historical record justifies) patenting as an urban phenomenon (31) and focus on the growth of patenting output in a single city. The assumptions of the model are:

1. Population of the city grows exponentially at a rate $g$, $N(t) = N_0 \mathrm{e}^{gt}$.

2. Inventors disproportionately concentrate in larger cities, and the number of inventors $I(t)$ per year follows the relation $I(t) = I_0 N(t)^\beta$, where $\beta > 1$.

3. The number of new patents produced by all inventors alive at that moment in the city in small interval (e.g., a year, $\Delta t = 1$ year) is proportional to all pair-wise interactions between inventors, $y(t + \Delta t) = \Delta x(t) = cI(t)^2 \Delta t$, where $c$ is a fixed proportionality constant.

The solution to these assumptions for how patents will accumulate with time in that is

$$
\begin{aligned}
x(t) &= \int_{t_0}^{t} c \left( I_0 N_0^\beta \mathrm{e}^{\beta g t'} \right)^2 \mathrm{d}t' \\
&= \left( \frac{c I_0^2 N_0^{2\beta}}{2\beta g} \right) \left( \mathrm{e}^{2\beta g t} - \mathrm{e}^{2\beta g t_0} \right). \qquad (15)
\end{aligned}
$$

As is evident, population growth in this model drives the growth of patents. Urban agglomeration of inventors and their interactions only change the rate in the exponential growth, which in this case becomes $\Delta x(t)/x(t) = r = 2\beta g$. It is thus difficult to explain the sub-exponential increase of patents in the US patent record and the transitioning to periods of super-exponential growth with urban agglomeration effects.. One could get a transition from exponential to super-exponential if one includes *all* the possible orders of interactions. Alternatively, super-exponential growth can also emerge by aggregating the contributions of all cities to the national level, for example, assuming that the rate of population growth is fixed for each city but it is drawn from a normal distribution across cities, $g_c \sim \mathcal{N}(\mu_g, \sigma_g^2)$ (proving that this leads to super-exponential growth at the aggregate national level is left to the reader as exercise).

We readily acknowledge the possibility that a complex mixture of population growth, spatial agglomeration of inventors, globalization, and historical contingencies on the evolution of the economy and the institutions supporting inventive activities, may explain the patterns we observe. In other words, there is no reason to expect that a simple model will faithfully reproduce real data. Notwithstanding this possibility, we limit our analysis to our unabashedly simple model of recombination of patents which, while abstracting away many of these aspects, still is able to reproduce the data surprisingly well.

# Appendix C. Mathematical steps

Here we show how to express the model in continuous time.

Let the "recently added" patents be $y(t_n) = x(t_n) - x(t_{n-1})$. In this way, the equation becomes

$$y(t_{n+1}) = py(t_n)x(t_{n-1}).$$

Subtracting from both sides $y(t_n)$,

$$y(t_{n+1}) - y(t_n) = py(t_n)(x(t_{n-1}) - 1).$$

Assume $t_{n+1} = t_n + h$ and let $h \to 0$, such that $y(t) = \mathrm{d}x(t)/\mathrm{d}t = \lim_{h\to 0} \frac{x(t_{n+1})-x(t_n)}{h}$, and $\mathrm{d}y(t)/\mathrm{d}t = \mathrm{d}^2x(t)/\mathrm{d}t^2 = \lim_{h\to 0} \frac{y(t_{n+1})-y(t_n)}{h}$.

Rearranging, we get

$$\begin{aligned} \frac{\mathrm{d}y(x)}{\mathrm{d}x}\frac{\mathrm{d}x(t)}{\mathrm{d}t} - cy(t)(x(t)-1) &= 0, \\ y(t)\left(\frac{\mathrm{d}y}{\mathrm{d}x} - c(x(t)-1)\right) &= 0. \end{aligned} \tag{16}$$

Hence, there is a trivial solution coming from the term $y(t) = 0$, which gives $x(t) = x_0$, a fixed and constant number of patents.

The other solution comes from solving the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - cx + c = 0. \tag{17}$$

The solution is

$$y(x) = \frac{1}{2}\left(cx^2 - 2cx + 2y_0\right),$$

where $y(x(0)) = y_0$. The next equation we thus need to solve is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2}\left(cx(t)^2 - 2cx(t) + 2y_0\right). \tag{18}$$