Invention as Cultural Accumulation

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Abstract

What type of growth process is the increase in inventive activity? Is invention a process of collective learning akin to cultural accumulation? What would be the quantitative signal in the record of patenting activity that would imply a process of cultural accumulation, as opposed to a growing historical record of differentiated inventions driven by market profitability? We propose that if patents accumulate simply because there is an increasing number of inventors patenting, then the accumulation is driven by scale, not by learning. Given this, can we disentangle the effects of scale and the effects of individual learning from the effects of collective learning in the data?

What is the evidence that patents represent a historical record designed to foster collective learning, as Thomas Jefferson imagined it to be? We investigate this question not from the point of view of the efficiency of market forces to foster innovation or prevent externalities, or from the point of view of how appropriate is patent law for incentivizing invention. Instead, we investigate this question from the point of view that considers the humans species as uniquely characterized by their capacity to collaborate and learn collectively. This view states that humans owe their ecological success to culture (defined as the body of information that has accumulated through a process of collective learning over many generations). Thus, our question is thus whether invention is a process of cultural accumulation, and whether this can be detected in the history of the patent record.

We highlight the fact that not all processes of accumulation in socioeconomic systems are akin to processes of cultural accumulation. For example, a trivial example is the accumulation of weeks in our calendars. This accumulation is just the consequence of social constructions we use to quantify time. Another example is the accumulation of CO_2 in the atmosphere, caused by human activity, yet due not to a process of collective learning but to a process of uncontrolled entropy creation. Similarly, the list of countries that exist worldwide has been growing, yet this accumulation of names in a list is due to various geopolitical processes, not learning. Our task is to investigate whether we can distinguish a simple process of accumulation from a complex process of collective learning, in order to answer the question of whether invention is, in fact, a process of cultural accumulation.

As is custom, we use patents as a proxy for invention, and we take the accumulation of patents in time as our main variable of interest. While it can be argued that the fact that inventors patent in teems and that the patents they invent cite other previous patents is *the* evidence that invention is a cultural process of collective learning, we argue that this stands as a precondition that makes the question worth asking. Furthermore, we argue that evidence of cultural accumulation must take the form of quantitative statistical patterns specific to how knowledge accumulates and builds on itself through a process of collective learning.

Our contribution consists of two main findings. First, we find that the microstructure of stochastic fluctuations in the time series of total number of patents displays the presence of memory, whereby new inventions have an enduring multiplicative effect on all subsequent rates of patent growth. And second, we find that the macropatterns of growth of the inventive activity are consistent with a model of recombination of previous inventions.

We believe our results are interesting for three reasons. On the one hand, these results bolster the idea that the patent system is yet another instance that humans have devised to coordinate their actions in order to act as a collective brain (Muthukrishna and Henrich, 2016). In other words, we find in patents evidence that invention is indeed a process of cultural accumulation. Next, these results in turn imply that the US patent system does indeed accomplish the mission codified in article 112 of the US Patent Code that states that "patents must clearly *disclose* and *teach*". Finally, as will become clear from our detailed analysis below, we conclude that processes of collective learning cannot be identified, in general, through a single kind of growth (e.g., exponential). Given the significance our results place on culture and collective learning as fundamental aspects of inventive activities, has the role of incentives that economic thinking has directed towards individuals been misguided?

I. Background

Processes of continuous, open-ended, growth seem to be a defining feature of the human species, in contrast to processes in non-human species. Which other animal has grown in population and ecological extension so much as ours, or is as diverse? Given the remarkable differences in scale and diversity of social systems as compared to other animals, a central question in the field of human evolutionary biology has been why, to begin with, do social systems grow in size and diversity *at all*?

Work in the field of cultural evolution suggests that growth in socioeconomic systems is really a process of accumulation. The claim is that, in humans, growth (in population, for example, but also in material wealth) is the consequence of the fact that adaptive information about how to cope with the environment not only is transmitted from one generation to the next, but that this information *accumulates*. And this accumulation propels demographic and socioeconomic growth (Ghirlanda and Enquist, 2007). While some animals have the capacity to transmit information from individual to individual, none have the capacity to accumulate this information over generations. Culture gets to be defined in this way as the body of norms, beliefs, ideas and knowhow that has accumulated over generations and is stored in people's brain.

At the basis of our capacity to collectively accumulate information over generations is (i) our individual capacity to copy and learn from others that are more knowledgeable than we are and (ii) our capacity to collaborate and do joint work with our peers. While the presence of imitation and collaboration leads to the accumulation of knowledge and growth, the reverse is not necessarily true. In other words, not all growth processes are processes of cultural accumulation. Does this imply there is a distinctive quantitative characteristic about growth that differentiates a process of cultural accumulation from growth that arises from a different process?

More than half a century ago Lehman (1947) provided the first thorough review of the accumulation of culture in different areas of knowledge such as philosophy, medicine, economics, and many others (using data that started in 1200AD for some areas). He concluded that accumulation in all these fields follows an exponential law. But is all cultural accumulation described by exponential growth? Enquist et al. (2008) posit that we are likely to observe exponential increases in the amount of culture if the creation of new culture is driven by culture itself. In other words, if culture begets culture.

In a subsequent publication, Enquist et al. (2011) show that the type of growth by which culture accumulates can change between linear, exponential, or super-exponential depending on how cultural traits are acquired. For example, if cultural traits are acquired independently

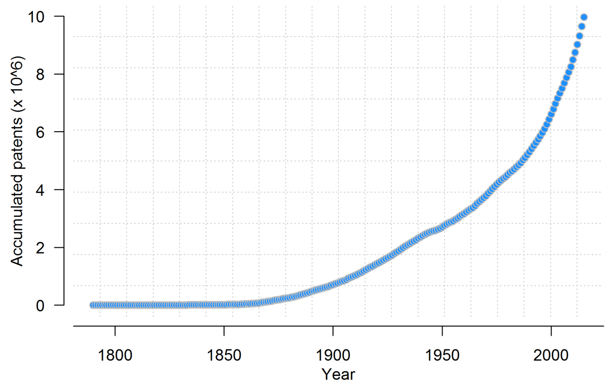

Figure 1: Is the accumulation of patents over the years a process of cultural accumulation?

from one another, culture will grow linearly. This process, however, should not be referred to as cultural accumulation since there is no sense in which a population of individuals would need to coordinate their actions in order for this accumulation to happen. In this sense, one has to distinguish *collective learning* from a collective of *individuals learning*. However, even if cultural traits are not acquired independently but rather they affect each other, culture can grow following different functional forms. Enquist et al. (2011) show that exponential growth occurs when accumulation is driven by differentiation, i.e., when an elements becomes two slightly different new versions (e.g., speciation). Interestingly, they also show that one can obtain super-exponential growth when accumulation is driven by recombination.

The question in this paper boils down to the following: is Figure 1, the time-series of the total number patents in the US patent system spanning more than two centuries, the result of a process of collective learning? Or, given that it is a historical record, is it just the time series of a trivial process of accumulation?

II. The characteristics of collective learning

We are going to propose that a reasonable proxy for quantifying knowhow is "the number of things you know how to do". It is how we assess in practice how much an individual knows. We ask people questions like "how many languages do you know how to speak?", "how many different mathematical problems do you know how to solve?", "how many cooking recipes do you know how to cook?".

To quantify collective know-how, we propose to count how many things a collective knows how to do. Now, this definition is in part problematic, because if all the things that a given society knows how to do, are simultaneously all things that each individual in that society also knows how to do, then what would we say is the collective knowhow of that society? Arguing there is, in fact, collective knowhow in that society would miss the point of the distinction between individual and collective knowhow. Let us define collective knowhow as the number of things a collective knows how to do, that no individual would know how to do.

Similarly, what would be a good definition of collective learning? Arguably, collective learning is the increase in collective knowhow that is not accounted by increases in individual knowhow. This would be in contrast with increases in educational attainment in a society, but in line with specialization and division of labor. Under this definition, an important channel for collective

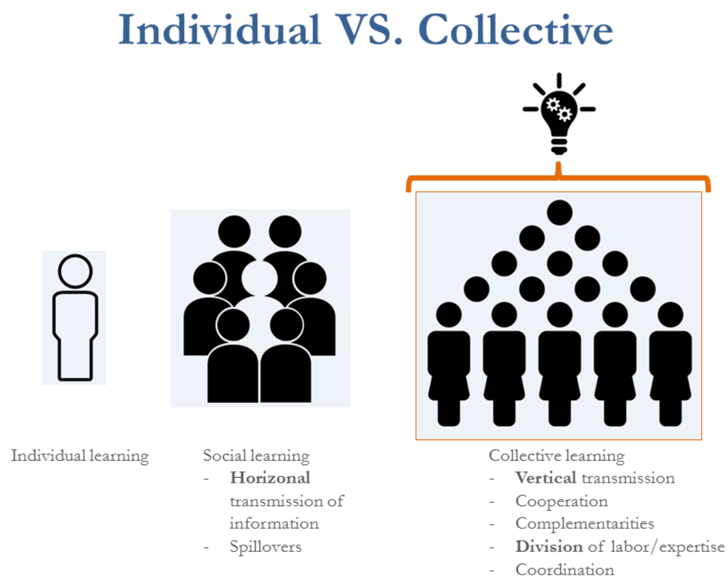

Figure 2: Differences between types of learning.

learning would be the incorporation of new practices or ways of doing things brought into a society by immigrants.

There is an extensive literature both in economics and cultural evolution that focuses on questions about how individuals learn. Do they learn from parents, in school, in their jobs, from random encounters in the city they live in? The convention is to hold a view in which individuals are in the middle of pathways where information flows, and the working hypothesis is that how much individuals will learn depends on where in these networks individuals are located in. By implication, individuals with high centrality in the network of information flow will tend to be the most inventive. This emphasis on information flow in economics is reflected in a very strong focus on externalities and knowledge spillovers. And in psychology and anthropology this focus is reflected in the emphasis on social learning as the characteristic that distinguishes humans from other animals. We want to argue that we need to think beyond social learning, and consider the situations in which individuals are not learning anymore, at least not significantly, and yet the group as a whole can still keep accumulating know-how that is instead embodied not in how much individuals know, but by their division of labor and their capacity to complement each other.

We want to make a distinction between three types of learning: individual learning, social learning, and collective learning. In Figure 2[1] , the first image to the left is the view of invention as the consequence of individual learning. The second, middle, image would be individual invention because of social learning. And the third image to the right is just invention as collective learning, and in this view, we can almost disregard the role of the individual. There is, obviously, one or a few authors behind an invention, but one can regard those as just the tip of the iceberg, if you like. Or as Robert Boyd says, cultural novelty comes from people standing on the shoulders of midgets.

[1] Images from Noun Project. On the left: Created by Ben Peetermans. In the middle: Created by Blake Thompson. On the right: Created by Wilson Joseph.

You still may wonder what is the distinction between this picture of invention as enabled by social learning, and this picture over here of invention as collective learning. That is partly the question, whether there is any useful distinction between these two. And if there is a distinction, would it be relevant for invention, and would they leave different traces and signals in the data.

The story of social learning emphasizes the horizontal transmission of information and the knowledge spillovers. But from the point of view of collective learning, the vertical flows of information across generations of people are more important, also the cooperation between people, and the fact that individuals can complement each other, all of which have to do with the division of labor. We posit that these two views actually paint very different pictures of the world.

The field in which this hypothesis has been taken more seriously is Cultural Evolution. From Muthukrishna and Henrich: "just as thoughts are an emergent property of neurons firing in our neural networks, innovations arise as an emergent consequence of our species' psychology applied within our societies and social networks. These can produce complex designs without the need for a designer–just as natural selection does in genetic evolution".

Finally, we want to argue that it is a "self-propelled" process of accumulation. Here is where the connection with processes of cultural accumulation are the clearest. If invention is a process of cultural accumulation, it implies that there is path dependency, which in turn means that past and present inventions will affect future inventions permanently. This, again, is something that has to some extent been studied in some of the work in cultural evolution, in which cultural traits can foster or inhibit the appearance of other cultural traits.

To summarize, a process of collective learning has three characteristics, which we take as hypothesis to be tested:

1. Accumulation must come from *collective efforts*. In the context of patents, it means that most inventions should come from building upon the knowledge that others have came up with, typically from collaborative efforts, rather than independent contributions of lonely inventors. In particular, the dominant way of patenting should be by recombination as opposed to novel origination.

2. The rate of accumulation must display *path-dependency*. In other words, the time-series of rates should have memory of past events.

3. The broad time series of accumulated patents must be consistent with a model where invention is (mainly) the result of recombining new with old ideas.

III. Results

III.1 Evidence of collaborative efforts

[HERE I THINK WE SHOULD REVIEW SOME OF THE RESULTS FROM THE LITERATURE, IN PARTICULAR DEBBIE'S PAPERS.]

III.2 The micro patterns of collective learning

If invention can be understood as a process of cultural accumulation, then patents are at the same time the input and the output of a process of collective learning. In this view, patents are not just the byproduct of a learning process used for commercial purposes. They are "tools for learning", and as such, a patent can in principle affect permanently the rate of future learning (i.e., the future rate of invention). Patents not only provide excludability rights, not only open new spaces in the

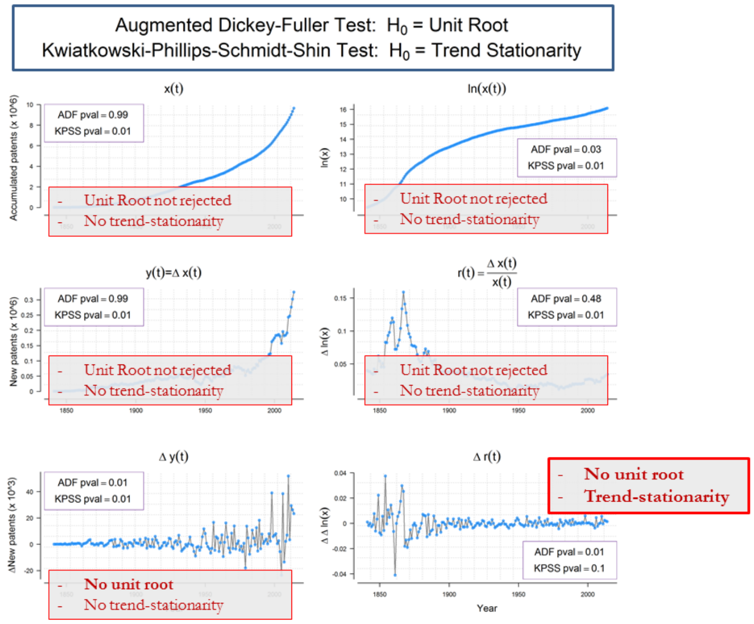

Figure 3: Testing unit-root and trend stationarity.

"adjacent possible"; They also teach us how to open the spaces precisely in the direction in which exploration can be faster.

The hypothesis we want to test is whether a stochastic shock in the rate of invention has an effect on the future rate of inventions that is permanent or transitory. This hypothesis can be tested through testing whether the time-series of invention has a *unit root*. The simplest stochastic process with a unit root is $x(t) = ax(t-1) + \varepsilon_t$, where $a = 1$. Notice that a way to test the hypothesis of a unit root, one can perform a regression of $\Delta x(t) \equiv x(t) - x(t-1)$ against $\beta_0 + \beta_1 x(t-1)$. If there is a unit root, then $\beta_1 = 0$, and $\widehat{\beta_1}$ should not be significantly different than zero. This intuition is what is behind the Dickey-Fuller test that we will use below, which essentially performs the test that, under the null-hypothesis, $\beta_1 = 0$.

In practice, it is hard is to reject the hypothesis of the unit root. Time series often pass the unit root simply because time series are typically not stationary. Thus, what we really want is to find, by differencing the series several times, the stage at which we can reject the hypothesis of the unit root. In parallel, we want in addition to test for stationarity.

Here we use the *augmented* Dickey-Fuller (ADF) test, which allows a more flexible time-series model than the usual DF test, and we use the Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test for trend stationarity. We are looking for the point in which I cannot reject the KPSS test and simultaneously I can reject the ADF test. In the last graph of Figure 3, we finally get to a point where we can reject the hypothesis of a unit root and cannot reject the hypothesis that this is a stationary series. There is this initial period in the patenting system in the middle of the nineteen century, but it appears that indeed, since 1900âĂŹs the series of the yearly change of the rate of growth of the accumulated patents is stationary. This is interesting, because by necessity, the series just above this one has a unit root, meaning that approximately the rate of change of patenting is the same as rate of change of the previous period, plus something that can be deterministic with some stationary noise.

III.3 The macro patterns of collective learning

As we observed in the last section, the time-series of patents has memory. If new patents are the outcome of the body of pre-existing patents, then the evolution of patenting has more to do with the size of that body, than with how much time has passed. In a sense, a process of collective learning should not be analyzed as time-series, but as a sequence. The "tempo" of invention is not time as measured in years, but rather as measured by inventions themselves. Thus, instead of plotting new patents each year as a function of time, let us plot the number of new patents as a function of the total number of accumulated patents up to that point.

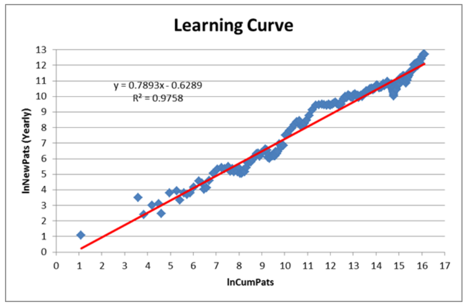

Figure 4: New patents each years as a function of the accumulated patents so far.

We observe in the patents data that in the US, in the aggregate, the number of new patents in a given year t is associated with the total number of accumulated patents until that moment through a power-law relationship:

$$\dot{x}(t) = \gamma x(t)^{\delta}, \tag{1}$$

where $x(t)$ is the accumulated number of patent until year t, $\dot{x}(t)$ is its time-derivative (i.e., the new patents in a relatively small interval dt, possibly like a year), and γ and δ are constants. Empirically, we get $\widehat{\delta} \approx 0.789$, which is shown in Figure 4. Note that this "elasticity" literally says that a 1% increase in the total body of patents as been associated with an increase in the number of new patents that are produced in the subsequent year of 0.8%.

Equation (1) can be solved in order to express x as a function of time,

$$\int_{x_0}^{x(t)} \frac{\mathrm{d}x}{x^{\delta}} = \int_{t_0}^{t} \gamma \mathrm{d}t,$$

yielding the general solution

$$x(t) = x_0 \left[1 + \left(\frac{\gamma - \gamma\delta}{x_0^{1-\delta}} \right) (t - t_0) \right]^{\frac{1}{1-\delta}} \tag{2}$$

which in turn has three solutions for different values of δ:

$$x(t) = \begin{cases} c_1 \left((x_0/c_1)^{1/\alpha_1} + (t - t_0) \right)^{\alpha_1}, & \text{for } \delta < 1 \text{ and } t > t_0, \quad \text{(3a)} \\ x_0 \, \mathrm{e}^{\gamma(t-t_0)}, & \text{for } \delta = 1 \text{ and } t > t_0, \quad \text{(3b)} \\ \dfrac{c_2}{(t_{\text{critical}} - t)^{\alpha_2}}, & \text{for } \delta > 1 \text{ and } t > t_0, \quad \text{(3c)} \end{cases}$$

where $\alpha_1 = 1/(1-\delta)$, $c_1 = (\gamma(1-\delta))^{\alpha_1}$, $\alpha_2 = 1/(\delta - 1)$, $c_2 = (\gamma(\delta-1))^{-\alpha_2}$, and $t_{\text{critical}} = t_0 + \gamma(\delta - 1)x_0^{\delta-1}$ is the critical time that defines the finite-time singularity when $\delta > 1$. It is

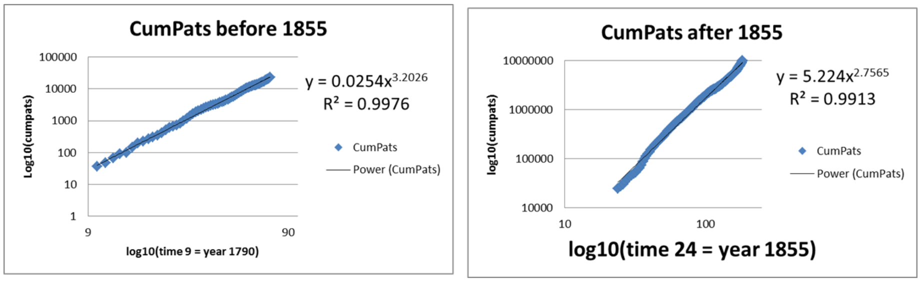

Figure 5: Accumulation of patents in time.

important to notice that in Equation (2) time is always compared with a specific year of reference t_0. The three equations describe very different types of growth in time. Equation (3a) is sub-exponential growth, Equation (3b) is exponential, and Equation (3c) is super-exponential growth. The latter implies that when $\delta > 1$ there will be a divergence of patents in a finite time.

From fig. 4 we observe that for the whole period of time the general trend is better described by the first solution, i.e., a sub-exponential growth (Equation (3)). The exponent that we estimate directly (plots in fig. 5) is $\widehat{\alpha} \approx 3$. Hence, while the growth is slower than exponential, it is still described by a superlinear growth function in the variable $t - t_0$.[2] To estimate t_0 a rigorous estimation is yet to be carried out. Manually, however, we see that there are two epochs: $t_0 \approx 1790$ and $t_0 \approx 1855$.

However, we also observe that there are small periods where there may be super-exponential growth. For that, we estimated the exponent in eq. (1) across different 50-year periods, from 1790 to 2015. We plot the results in Figure 6.

We observe in fig. 6 that there are four distinct periods of time in which the behavior switched between sub- and super-exponential eras. The accumulation of patents went (approximately) first through a process of sub-exponential growth between 1790 and 1837 ($\sim$ 50 years), then through a second period characterized this time by super-exponential growth between 1838 and 1869 ($\sim$ 30 years), then again between 1870 and 1960 ($\sim$ 90 years) patents accumulated sub-exponentially, and finally, since 1961 until today ($\sim$ 60 years), patents have been accumulating super-exponentially. In total, this means that there have been approximately 140 years of sub-exponential growth, and only 90 years of super-exponential growth. The overall dominance of the sub-exponential growth is the reason why fig. 6(C) has mostly a concave shape.

Thus, we need to develop a model that explains Equations (1) and (2), and find out whether these equations, their functional forms together with the values of the parameters, support the idea that the accumulation of patents is the result of a process of collective learning, or whether it is a simple process of independent accumulations. In particular, we would like to know whether a process of collective learning leads to sub- or super-exponential growth.

The elements that change in time and which we (possibly) need to model and take into account are:

$n(t)$: The total population size (e.g., working age population).

$y(t)$: The number of inventors within the (working age) population.

$q(t)$: The difficulty of "inventing" in time.

$x(t)$: The number of accumulated patents.

[2] If we use the empirically estimated values of Equation (1), we get $\widehat{\alpha} \approx 4.74$.

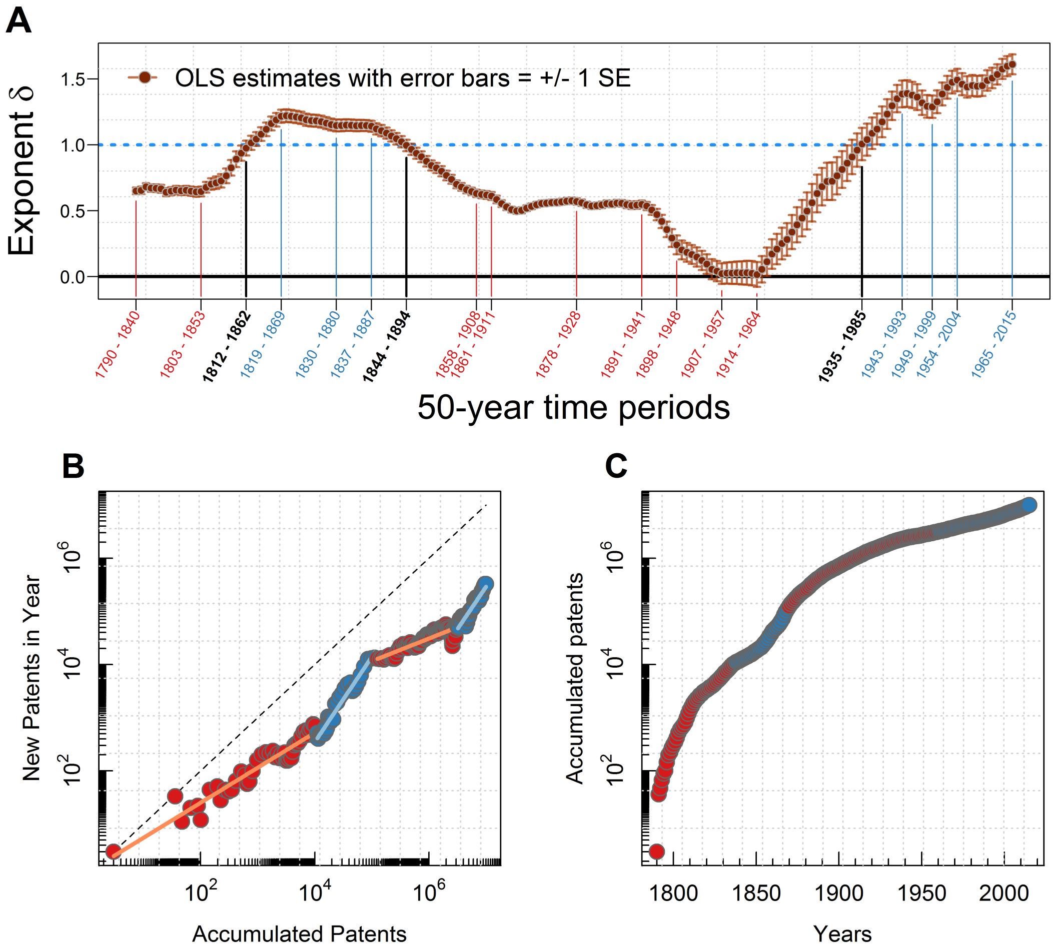

Figure 6: Whether patents accumulate slower, equal or faster than exponentially depends on the value of the exponent δ. **Top panel:** The graph shows the estimation of this exponent over many sequential running windows of 50 years, from 1790 until 2015. It is observed that the value changes in different periods of time. The colored vertical lines identify sudden changes in the behavior of the exponent over time, where red are those periods where growth was sub-exponential and blue super-exponential. The black vertical lines identify the periods in which there was a change in regime from sub- to super-exponential growth, or viceversa. **Bottom panels:** The plot on the left shows the scatter of the new patents as a function of the accumulated patents in log-log scales, and the plot on the right shows the accumulated patents as a function of time, with the y-axis shown in a log scale. In both plots, red dots represent the periods of time in which patents accumulated sub-exponentially, while blue dots represent periods of super-exponential growth.

IV. New inventions as combinations of old with current ideas

Here we will focus only on the combinatorial dynamics of patents. The assumptions of the model will be:

1. Patents are the only entity being modeled, and the number of patents is denoted by $x(t)$.
2. Patents at any given time can be divided in three: *old* patents, *recently added* patents, and *new* patents.
3. Evolution will happen such that the *new* patents are the pair-wise combinations of the *recently*

added patents with the *old* patents. In other words, a combination of an old patent with a recently added patent results in a single novel patent.

As an example, the accumulation of patents may proceed as follows:

1. At t_0, start with $x(t_0) = 2$ as the initial number of patents. Just for convenience, let us label one of the patents as a and the other one as b.

2. In the next time-step $t_1 = t_0 + 1$, there is a single combination of patents that can be done with the existing pair of patents from t_0, a and b, which we will label as $a + b \longrightarrow ab$. Thus, $x(t_1) = x(t_0) + 1 = 3$, because we now have in the record the patents a, b and ab.

3. In the next time-step $t_2 = t_1 + 1$, even though the third patent ab was itself a combination of the original two patents, it can still get *re*-combined with each of the old separately. Hence, there are two new recombinations, $a + ab \longrightarrow aab$ and $b + ab \longrightarrow bab$. At t_2, then, we have $x(t_2) = x(t_1) + 2 = 5$, because now in the record the following patents appear: *old*$= \{a, b\}$, *recently added*$= \{ab\}$, and *new*$= \{aab, bab\}$. Note, however, we do not allow a and b to recombine again, since that possibility was already exhausted.

4. In the next time-step $t_3 = t_2 + 1$, there are two patents that were recently added. These can be recombined with the old three patents ($x(t_1) = 3$) separately. Thus, $x(t_3) = x(t_2) + 2 * 3 = 11$. The list of patents at this point is: *old*$= \{a, b, ab\}$, *recently added*$= \{aab, bab\}$, and *new*$= \{aaab, abab, baab, bbab, abaab, abbab\}$.

5. And so on...

In general, at next time-step t_{n+1}, there are $x(t_n) - x(t_{n-1})$ patents that are the *recently added*, which can get recombined with the *old* patents $x(t_{n-1})$. Therefore, the number of patents at the end of time-step t_{n+1} is

$$\overbrace{x(t_{n+1})}^{\text{next}} = \overbrace{x(t_n)}^{\text{current}} + \underbrace{\underbrace{x(t_{n-1})}_{\text{old patents}} * \underbrace{(x(t_n) - x(t_{n-1}))}_{\text{recently added patents}}}_{\text{recombination}}.$$

One can generalize a bit the model and assume

(i) that a fraction $1 - p_1$ of the old patents are forgotten or obsolete and hence cannot be used for recombinations, which means that only a fraction p_1 of old patents at any given point in time are recombinable,

(ii) that there are some constraints from the "specificity" or "specialization" of patents that prevents the new patents from combining with the old patents, and thus only a proportion p_2 of the recently added patents can be combined with the old ones,

(iii) and finally, we can assume not all possible recombinations are discovered but only a fraction p_3.

Adding these assumptions yields

$$\begin{aligned} x(t_{n+1}) &= x(t_n) + p_3[p_2(x(t_n) - x(t_{n-1}))][p_1 x(t_{n-1})], \\ &= x(t_n) + p(x(t_n) - x(t_{n-1}))x(t_{n-1}) \end{aligned} \tag{4}$$

where $p = p_1 p_2 p_3$, and where the initial conditions, following the example, would be $x(t_0) = 2$ and $x(t_1) = 3$.

IV.1 Continuous time solution

There is no analytic solution for Equation (4). Equation (4) can be re-expressed in continuous time, which can lead to a more tractable differential equation. Hence, let the "recently added" patents be $y(t_n) = x(t_n) - x(t_{n-1})$. In this way, the equation becomes

$$y(t_{n+1}) = py(t_n)x(t_{n-1}).$$

Subtracting from both sides $y(t_n)$,

$$y(t_{n+1}) - y(t_n) = py(t_n)(x(t_{n-1}) - 1).$$

Assume $t_{n+1} = t_n + h$ and let $h \to 0$, such that $y(t) = \mathrm{d}x(t)/\mathrm{d}t = \lim_{h\to 0} \frac{x(t_{n+1})-x(t_n)}{h}$, and $\mathrm{d}y(t)/\mathrm{d}t = \mathrm{d}^2x(t)/\mathrm{d}t^2 = \lim_{h\to 0} \frac{y(t_{n+1})-y(t_n)}{h}$. Thus, in continuous time, we write the model as

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = cy(t)(x(t) - 1), \tag{5}$$

where $c = p/h$. Rearranging, we get

$$\begin{aligned}
\frac{\mathrm{d}y(x)}{\mathrm{d}x}\frac{\mathrm{d}x(t)}{\mathrm{d}t} - cy(t)(x(t) - 1) &= 0, \\
y(t)\left(\frac{\mathrm{d}y}{\mathrm{d}x} - c(x(t) - 1)\right) &= 0.
\end{aligned} \tag{6}$$

Hence, there is a trivial solution coming from the term $y(t) = 0$, which gives $x(t) = x_0$, a fixed and constant number of patents.

The other solution comes from solving the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - cx + c = 0. \tag{7}$$

The solution is

$$y(x) = \frac{1}{2}\left(cx^2 - 2cx + 2y_0\right),$$

where $y(x(0)) = y_0$. The next equation we thus need to solve is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2}\left(cx(t)^2 - 2cx(t) + 2y_0\right). \tag{8}$$

The solution is

$$x(t) = 1 + \frac{K}{c}\tan\left(\frac{K}{2}t + \tan^{-1}\left(\frac{x_0 - 1}{K/c}\right)\right), \tag{9}$$

where $K = \sqrt{2y_0c - c^2}$ and $x_0 = x(0)$.

IV.1.1 The tangent growth function

The *tangent* function has interesting properties and predicts an interesting behavior for the growth of patents. To see the properties more clearly, let us introduce the "critical time" t_{critical}:

$$x(t) = 1 + \frac{K}{c}\tan\left(\frac{\pi}{2} - \frac{K}{2}(t_{\text{critical}} - t)\right), \tag{10}$$

where

$$t_{\text{critical}} = \frac{\pi}{K} - \frac{2}{K}\tan^{-1}\left(\frac{x_0 - 1}{K/c}\right).$$

At the critical time, the number of accumulated patents would diverge, $x(t \to t_{\text{critical}}) \to \infty$, and thus t_{critical} defines the time of a finite-time singularity.

To better understand the transition from sub-exponential growth to super-exponential growth, we use the fact that the tangent function has a simple series expansion in terms of odd-order polynomials

$$\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \ldots,$$

for all $|z| < \pi/2$. In the our case,

$$\begin{aligned} x(t) &= 1 + (K/c)\tan(z_t), \\ &= 1 + (K/c)z_t + \frac{(K/c)}{3}z_t^3 + \frac{2(K/c)}{15}z_t^5 + \ldots, \end{aligned} \tag{11}$$

where the parameter of the tangent is a function of time,

$$z_t = \frac{\pi}{2} - \frac{K}{2}(t_{\text{critical}} - t).$$

The series expansion implies, for example, that when $z \approx 0.1$ the non-linear terms are on the order of 10^{-4} or smaller, and thus growth is linear. If $z \approx 0.2$, then the cubic term becomes of the order of 10^{-3} and the quintic term is of the order now of 10^{-4}.

One can observe very clearly from Equation (11) that the accumulation of patents, according to this model, goes from linear (as t), to super-linear (as t^3), and eventually to super-exponential.

IV.2 Empirical estimation

Here I simply estimate the following equation:

$$x(t) = 1 + A\tan(Bt + \phi). \tag{12}$$

I will refer to this model as "Model 4".

I then compare this fit with a simpler model with cubic terms and also three parameters:

$$x(t) = A + B(t - C)^3. \tag{13}$$

I will refer to this model as "Cubic Model".

The estimation of these models is done to the data post-1870, since that date is after the short period of super-exponential growth. Figure 7 shows the fits of both models, putting the y-axis in logarithmic scales. According the the estimated values of the parameters of Equation (12) we can recreate the parameters of Equation (10), in particular the critical time. For that we get the year $\widehat{t_c} \approx 2051$.

V. Enquist et al. (2010)

In this paper, Enquist and collaborators show that in order for culture to be maintained individuals must be able to learn from not just their (biological) parents, but from many (cultural) parents. The probability of learning a cultural trait changes also if the learner has many opportunities, or trials.

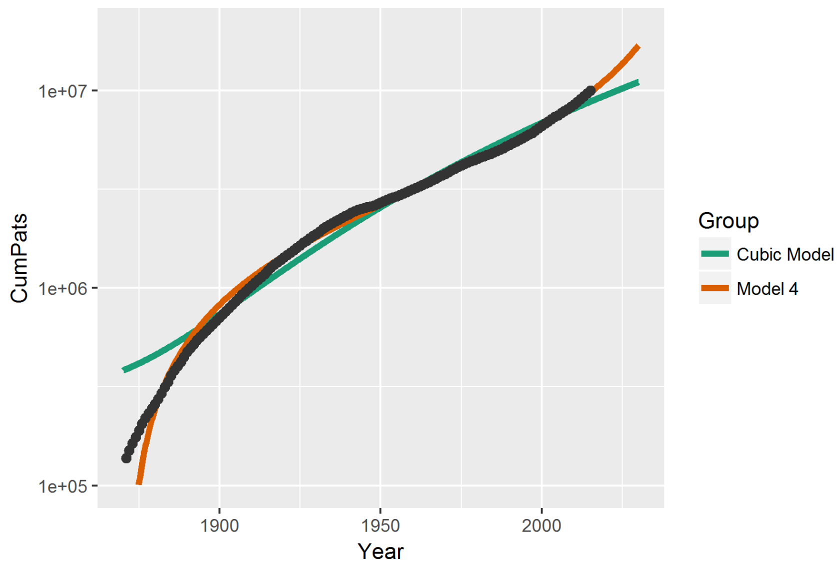

Figure 7: Lines are fits to the data. Both fitted models have three parameters. "Model 4" visually out-performs the "Cubic Model".

Table 1: The probability of learning a cultural trait under different conditions (Enquist et al., 2010).

	One cultural parent	Many cultural parents ($= m$)
One learning trial	pq	$1 - (1 - pq)^m$
Many learning trials ($= n$)	$(1 - (1 - p)^n)q$	$1 - (1 - (1 - (1 - p)^n)\, q)^m$

The results are shown in Table 1. There, q represents the probability that a cultural parent has the trait, and p represents the probability that learning the trait is successful in a single learning trial.

The "many learning trials" results are, in fact, equivalent to the "one learning trial" results, where the equivalence is $p \leftrightarrow 1 - (1 - p)^n$. The point here is that the model that comes out of this is:

$$x_{t+1} = 1 - (1 - px_t)^m. \tag{14}$$

In Enquist et al. (2010) words,

> [...] changing the model so that individuals have more than one cultural parent has a dramatic effect on the possibility of longstanding culture. In this case, if innovations made by a single individual are lucky enough to spread in the first few generations, then they can be maintained in the population for a long time by social learning alone.

VI. Simple models

VI.1 Model #1: emphasizing population growth

In this model the only driving force is population growth: more population, more inventors, more patents.

1. Population grows exponentially in time at rate r, $N(t) = N_0 \mathrm{e}^{rt}$.
2. Inventors are a constant fraction of the population, $I(t) = I_0 N(t)$.
3. Each inventor patents at a rate μ, meaning that in a small interval of time Δt, a single inventor produces, on average, $\mu \Delta t$ patents. Hence, the total number of patents produced by all inventors alive at that moment in that small interval is $\Delta x = I(t)\mu\Delta t$.

Putting together all the assumptions and solving the model yields

$$
\begin{aligned}
x(t) &= \int_{t_0}^{t} I_0 N_0 \mathrm{e}^{rt'} \mu \mathrm{d}t' \\
&= \left(\frac{I_0 N_0 \mu}{r} \right) \left(\mathrm{e}^{rt} - \mathrm{e}^{rt_0} \right). \qquad (15)
\end{aligned}
$$

This model generates exponential growth in the number of patents simply because total population is growing exponentially.

VI.2 Model #2: emphasizing urbanization and the agglomeration of inventors

In this model we introduce the fact that as population grows, it concentrates in larger cities, which themselves disproportionately concentrate inventors. We model only one city below, as aggregating over many cities would "dilute" the agglomeration effect.

1. Population grows exponentially in time at rate r, $N(t) = N_0 \mathrm{e}^{rt}$.
2. Inventors are an increasing fraction of the population, $I(t) = I_0 N(t)^\beta$, where $\beta > 1$.
3. As in Model 1, each inventor patents at a rate μ, meaning that in a small interval of time Δt, a single inventor produces, on average, $\mu \Delta t$ patents. Hence, the total number of patents produced by all inventors alive at that moment in that small interval is $\Delta x = I(t)\mu\Delta t$.

Putting together all the assumptions and solving the model yields

$$
\begin{aligned}
x(t) &= \int_{t_0}^{t} I_0 N_0^\beta \mathrm{e}^{\beta rt'} \mu \mathrm{d}t' \\
&= \left(\frac{I_0 N_0^\beta \mu}{\beta r} \right) \left(\mathrm{e}^{\beta rt} - \mathrm{e}^{\beta rt_0} \right). \qquad (16)
\end{aligned}
$$

This model *also* generates exponential growth in the number of patents (also because of population exponential growth). The difference, however, is that the rate of patent growth is larger than the rate of population growth, driven by the fact that inventors appear at higher and higher rates in larger cities. Moreover, since we know that "more complex" technologies have scaling exponents β that are larger relative to "less complex" technologies, the model would predict that the accumulation of "more complex" patents is faster.

In any case, this model does not show what we observe in the sense that patents have been growing for the most part sub-exponentially.

VI.3 Model #3: emphasizing that patents arise from interactions between inventors

In this model we change the assumption that each inventor patents at a constant rate. We assume that the new patents come from interactions between inventors. The assumptions are:

1. Population grows exponentially in time at rate r, $N(t) = N_0 e^{rt}$.
2. Inventors are an increasing fraction of the population, $I(t) = I_0 N(t)^\beta$, where $\beta > 1$.
3. The total number of new patents produced by all inventors alive at that moment in a small interval Δt comes from, and is proportional to, pair-wise interactions, $\Delta x = \mu I(t)^2 \Delta t$.

The solution is

$$\begin{aligned} x(t) &= \int_{t_0}^{t} \mu \left(I_0 N_0^\beta e^{\beta r t'} \right)^2 \mathrm{d}t' \\ &= \left(\frac{\mu I_0^2 N_0^{2\beta}}{2\beta r} \right) \left(e^{2\beta r t} - e^{2\beta r t_0} \right). \end{aligned} \tag{17}$$

As is evident from the last three models, none of the effects of population growth, the urban agglomeration of inventors, nor interactions between the number of inventors alive can neither explain the sub-exponential increase of patents in the US patent record, nor the periods of super-exponential growth. From the functional form derived in the 3rd model, one could get super-exponential growth if one were to include all the possible orders of interactions. That is, not only the pair-wise interactions, but also combinations between three inventors, four, and so on. In that way,

$$\begin{aligned} x(t) &\propto \sum_{k=0}^{\infty} \frac{\left(\mu e^{\beta r t} \right)^k}{k!} \\ &\propto \exp\left[\mu e^{\beta r t} \right]. \end{aligned} \tag{18}$$

Assuming that interactions of all orders need to be present to explain the super-exponential growth seems like a strong assumption.

At this point, we need a mechanism that can slow down the growth of patent creation such that the accumulation becomes sub-exponential, but we also need a simple mechanism that can lead to super-exponential growth.

VII. Conclusions

The word "patent" comes from the latin word "patere", which means "to lay open". In this way, patents have the mission of disclosing new practical knowledge clearly enough that others can use it and build upon it. Therefore, it is presumed the patent record is a coordinating institution that promotes the accumulation of knowhow, through the same mechanisms by which culture itself accumulates. Invention is nothing more than a process of cultural accumulation. Our paper stands as a contribution for presenting empirical evidence of this fact.

VIII. To Do's

From meeting 2017-08-18:

1. Patents serve the purpose of a teaching instrument for society (article 112).
2. Should we focus on the US? Jose argues that we should look at the full extent of the system, including all countries.
3. we want to count new unique "functionalities".
4. Debbie: What is the unit that we ought to count?
5. Redundancy.
6. Should we count the accumulation of the number of unique combinations.
7. see the accumulation of the unique number of inventors.
8. Does the sum of S curves gives the curve we observe?

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